Statistical Physics[1]

Prof. Sigrist, FS 2013
05/02/2014, 15:30-16:00, Sylvain DE LESELEUC

Summary In Prof. office, exam on the table, prof and assistant are sitting at the table

Description of the content: 1) Phonons (chapter 2) 2) Gas of Bosons, Bose-Einstein Condensation (chapter 2 and 3), Superfluids.

Ablauf We began by a little chat (where I am from, which lectures did I attended), then we started the exam with the first subject : phonons for me.

M.S What can you tell about phonons in a solid ?

S.dL So if one take a crystal for example, the atoms are all sitting around equilibrium positions in an harmonic potential. If we look at the excitations of the systems and do some calculations, we will see that they behave as a gas of non-interaction bosons, we call these excitations phonons. What's make phonons singular, is the finite number of degree of freedom in our solid, so if we look at the density of states (DOS) of phonons, there will be a cut-off at a given frequency : the Debye frequency. (I then draw a curve to illustrate it, but I give it a shape of a square root, instead of a square. The correct shape is illustrated on 1).

M.S Can you write the equation of the DOS ?

S.dL (A bit annoyed, because I don't know it by heart and will thus have to derive it again. But the professor helped me to find it again, correcting me when I was doing some errors. The full derivation is on the script p41) We have $D(\omega) \propto k^2$. (At that time I corrected my sketch and made it look like the one on 1)

M.S Ok, now can you tell me more about the specific heat of the system ?

[1] MSc Physics, CoreSubjects

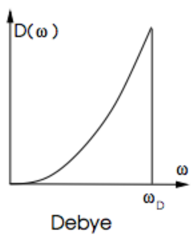


Figure 1: Density of states (DOS) of phonon in a solid

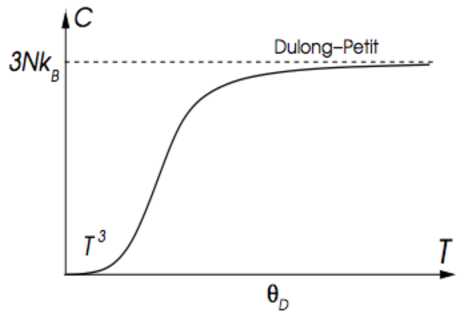


Figure 2: Heat capacity

S.dL At low temperature, C_V follows a T^3 law. At high temperature, it follows the Dulong-Petit law and converges toward to $3Nk_B$ where 3N is the number of degree of freedom of our system. (see 2)

M.S How would you prove it ?

S.dL First, I would calculate the evolution of energy of our system with temperature and then use it to find $C_V = \frac{dU}{dT}$. In order to find the energy, I will go back to the definition : $U = \int \mathrm{d}\epsilon \epsilon D(\epsilon) f(\epsilon)$ where f is the Bose-Einstein distribution and D the DOS of phonons, $\epsilon = \hbar\omega$ is the energy of a phonon. I will drop any pre-factor constant and only look at temperature behavior.

$$U \propto \int_0^{w_D} \mathrm{d}\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \tag{1}$$

For low enough temperature, we can forget about the Debye cut-off ω_D, and thus doing the change of variable $x = \beta\hbar\omega$, we obtain $U \propto T^4$. Then, we recover the law $C_V \propto T^3$at law temperature.

M.S What happens at higher temperature ?

S.dL Then we have to take care of the Debye cut-off, it will have a saturation effect and explains why C_V remains finite at infinite T. In this case, we can approximate $e^{\beta\hbar\omega} \approx 1 + \beta\hbar\omega$ and thus :

$$U \propto \int \mathrm{d}\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \propto \int \mathrm{d}\omega\omega^2 \frac{k_B T}{\hbar} \tag{2}$$

Eventually, we find $C_V \longrightarrow constant$.

M.S What others systems show the same behavior at low temperature ?

S.dL Photons for example.

M.S Ok, that's right. In fact, you only need a linear dispersion curve for low wave-vector. What other famous systems do show such a dispersion curve ?

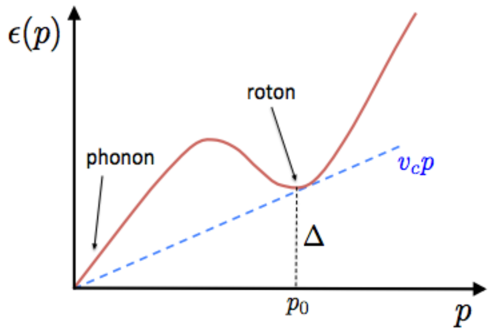


Figure 3: Dispersion curve of He4

S.dL Superfluids are an example. I then plotted 3 which is the dispersion curve of He4. In this case, we also have other excitations called rotons. In practice, they will lower the critical velocity below which one you cannot create any excitations.

M.S Bose-Einstein condensate are also an example of superfluid, what do you about single particle correlation in this system ?

S.dL (I plot 4) So first, let's see what happens in a thermal gas (with no condensate fraction). In this case, the correlation function decreases exponentially with the distance. If now we only have a BEC, the correlation function is constant. If now, increase the temperature, the correlation function will slightly decrease at the beginning, but will eventually goes to a finite value at infinity.

M.S How would you prove it ?

S.dL So, we use the definition of the single particle correlation function :

$$\frac{n}{2} g(r-r') = \langle \psi^\dagger(r)\psi(r')\rangle = \sum_{k,k'} e^{-ikr} e^{ik'r'} \langle a_k^\dagger a'_k \rangle = \sum_k e^{-ik(r-r')} \langle n_k \rangle \tag{3}$$

We see that if we have a thermal gas, $\langle n_k \rangle$ will be describe by the Bose-Einstein distribution, and the sum will always average to zero for big enough r-r'. Now, if we have a finite condensate fraction, it will give a finite contribution to the correlation : $g(\infty) = n_0$

M.S How does the condensate fraction evolves with temperature ?

S.dL (I plot 5) If we have no interactions and consider free bosons, it evolves with a critical exponent 3/2.

M.S We will prove it. To begin can you calculate the critical temperature ?

S.dL We look at the number of thermally excited particles $N_t h$, when it equals the number of particles for $\mu = 0$, we begin to have condensation.

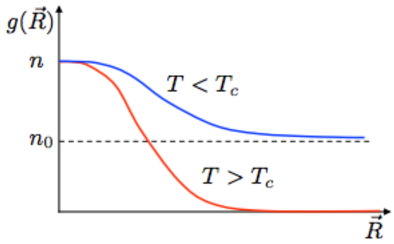


Figure 4: Correlation in a BEC

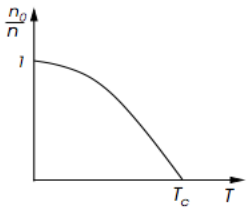


Figure 5: Condensate fraction of a BEC

$$N_{th} \propto \int \mathrm{d}\epsilon \frac{\epsilon^2}{e^{\beta\epsilon} - 1} \tag{4}$$

We see that if we decrease the temperature, the denominator increases and thus the number of thermally excited particles decreases. At some point, when it becomes less than the number of particles in our system, we have to put the difference in the ground-state -> it is Bose-Einstein condensation.

Now if we look at the temperature dependence, we can once again do the change of variable $x = \beta\epsilon$ and find $N_{th} \propto T^{3/2}$.

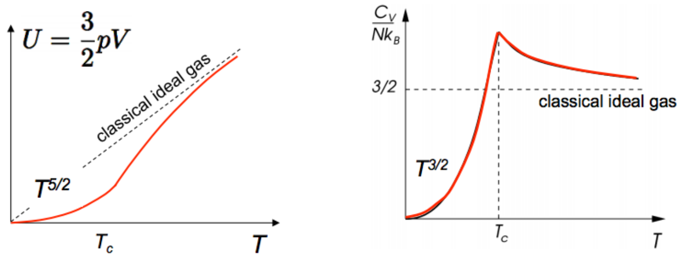
M.S What will be the heat capacity dependence of a Bose-Einstein condensate ?

S.dL I draw 6.a. For low T, we have $C_V \propto T^{3/2}$, which come from the energy being proportional to $T^{5/2}$. The cusp in the heat capacity indicates the phase transition towards the Bose-Einstein condensate phase.

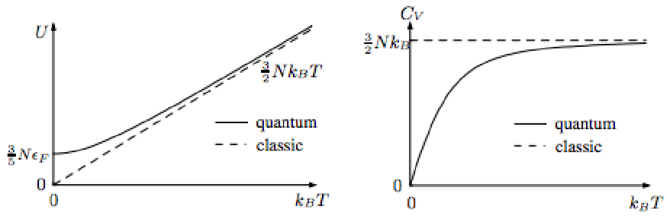
M.S What will it look like for a Fermi gas ?

S.dL I plot 6.b. At low T, the energy goes to a finite value with a quadratic dependency which is related to the heat capacity being linear (for low T).

M.S Ok, very good. That's all for the exam ! Enjoy your holidays !



(a) Bose gas



(b) Fermi gas

Figure 6: Energy and heat capacity of a Bose and a Fermi gas

Final Remarks The professor expects you to be able to do some calculations. You don't need to any pre-factors or every constants, but you should know the main idea of the famous derivation we did during the lectures. Also, when you do a plot, always write the axis, he was always asking for it because I was skipping it. He likes accuracy a lot, as you might have noticed during the lectures.

Expected mark: 5
Received mark: 5.75